Tentamen: Introduction to Plasma Physics

June 30, 2008

14.00-17.00 h, Room 5114.0004

Please write clearly your name on *each* sheet, and on the first sheet also your student number, date of birth, and address. You can use either Dutch or English according to your taste.

PROBLEM 1 (25 points)

A Penning trap is used to trap low-energy electrons or ions and consists of a quadrupolar electric field for axial confinement and a uniform magnetic field for radial confinement. The electric field is produced with a ring electrode and two endcap electrodes and is defined via an electrostatic potential function $\Phi(x, y, z)$ given by

$$\Phi(x, y, z) = \frac{\Phi_0}{2r_0^2}(x^2 + y^2 - 2z^2)\,. \quad (1)$$

The two endcap electrodes are negatively biased with respect to the ring electrode when electrons are being trapped and positively biased when trapping ions. The magnetic field is uniform and directed along the z-axis, i.e. $\mathbf{B} = B_0\mathbf{e}_z$. A schematic drawing of a Penning trap is shown below.

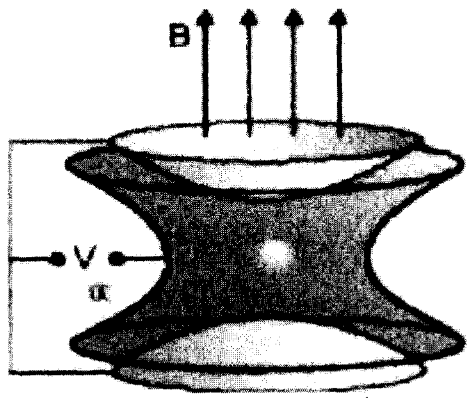

Figure 1: A Penning trap.

a. Discus qualitatively the single-particle motion of an electron in a Penning trap using the guiding-center approximation. Draw schematically the electron's trajectory. You may assume that the magnetic force is much stronger than the electric force.

b. Assume that an electron oscillates along the trap axis. Write down the equation of motion and derive a formula for the axial angular frequency ω_z.

c. Calculate the $\mathbf{E} \times \mathbf{B}$ drift velocity $\mathbf{v}_d$ and show that this drift leads to a slow rotation of the electron's guiding-center around the trap axis, the so-called magnetron motion. Calculate the angular frequency ω_- of the magnetron motion.

d. Write down the three Cartesian components of Newton's equation of motion of the electron moving in the electrostatic and magnetostatic fields of a Penning trap.

e. To solve for the radial motion derive a differential equation for the variable $u = x + iy$. Determine the two radial frequencies by inserting $u = u_0 e^{-i\omega t}$ in this differential equation and solving the resulting quadratic equation.

PROBLEM 2 (15 points)

a. *True or False* Debye shielding makes plasmas quasi-neutral on scale lengths much less than the Debye length. Motivate your answer.

b. *True or False* The parameter $\Lambda = n\lambda_D^3$, with λ_D the Debye length and n the density, is a large number in a well defined plasma. Motivate your answer.

c. Show that any distribution function $f(\mathbf{r}, \mathbf{v}, t) = F(H)$ with $H = m\mathbf{v}^2/2 + q\Phi$ solves the steady state $(\partial/\partial t = 0)$ collisionless Boltzmann equation.

PROBLEM 3 (20 points)

The effect of collisions can be included in the dispersion relation for waves in a cold plasma by adding a drag force $\nu_s m_s \mathbf{v}_s$ to the momentum balance equation, with ν_s the collision frequency and m_s the mass of the particle of type s:

$$m_s \frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}t} + \nu_s m_s \mathbf{v}_s = q_s \left(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}\right) \,. \tag{2}$$

a. Show that the effect of collisions can be obtained by making the substitution

$$m_s \rightarrow m_s \left(1 + \frac{i\nu_s}{\omega}\right)$$

in the collisionless dispersion relation.

b. For transverse waves in a cold plasma, show that if $\nu_s \ll \omega$ and $\omega_p \ll \omega$ (the high frequency approximation) the real and imaginary parts of the wave number are approximately

$$k_r = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{2\omega^2}\right) \,, \quad \text{and} \quad k_i = \frac{1}{2c}\sum_s \frac{\nu_s \omega_{ps}^2}{\omega^2} \,.$$

c. Find the dispersion relation for the longitudinal electron plasma oscillations including collisions. Briefly discuss the damping of these waves, i.e. what is the damping decrement and how does it depend on ν_c.

PROBLEM 4 (30 points)

A non-uniform plasma with equilibrium density $\rho_0(y) \propto \exp(y/s)$ is supported against the gravitational field $\mathbf{g} = -g\mathbf{e}_y$ by a magnetic field $\mathbf{B}(y) = B_0(y)\mathbf{e}_z$. Consider an interchange mode with a wave-like perturbation propagating in the x-direction as shown in figure 2(a). The plasma is bounded by conducting walls at $y = 0$ and $y = h$ as shown in figure 2(b).

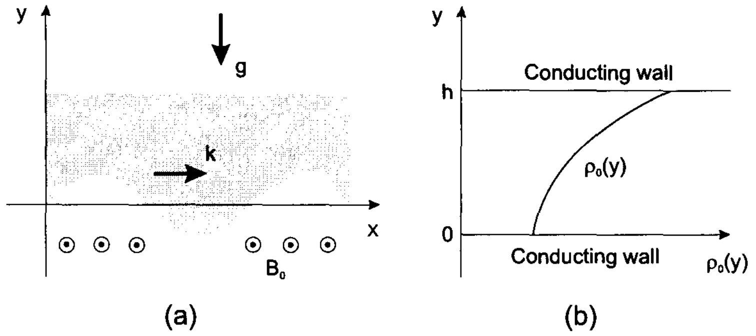

Figure 2: (a) A wave-like perturbation of a plasma that is supported against gravity by a magnetic field with straight field lines. The gravitational field $\mathbf{g}$ is in the $-y$ direction, the magnetic field $\mathbf{B}$ in the $+z$ direction and the perturbation propagates in the $+x$ direction. (b) The profile of the plasma mass density $\rho_0(y)$ between conducting walls at $y = 0$ and $y = h$.

a. Sketch in figures similar to figure 2(a) the direction of the electric fields generated by the drifting ions and electrons for both $s > 0$ and $s < 0$. Show also the direction of the resulting $\mathbf{E} \times \mathbf{B}$ drifts and indicate for both cases whether the interchange modes are stable or not.

b. Write down the ideal MHD momentum balance equation and show that the condition for magnetostatic equilibrium is given by $\mathrm{d}\mathbf{B}_0/\mathrm{d}y < 0$.

c. We will perform a linear stability analysis of the interchange instability. Show that from the first-order momentum equation it follows that

$$\nabla \times \left(\rho_0 \frac{\partial \mathbf{u}_1}{\partial t}\right) = \nabla \times (\rho_1 \mathbf{g}) \ . \tag{3}$$

d. Derive from this equation an ordinary first-order differential equation in y relating $u_x(y)$ and $u_y(y)$. (From now on the subscript 1 in $\mathbf{u}$ will be suppressed.)

e. Use the incompressibility condition to express $u_x(y)$ in terms of $u_y(y)$ and the linearized continuity equation to express $\rho_1(y)$ in terms of $\rho_0(y)$ and $u_y(y)$. Substitute these expressions into equation (3) and show that the following eigenvalue equation for $u_y(y)$ is obtained:

$$\frac{1}{\rho_0}\frac{\mathrm{d}}{\mathrm{d}y}\left(\rho_0 \frac{\mathrm{d}u_y}{\mathrm{d}y}\right) - k^2\left(1 + \frac{g}{s\omega^2}\right)u_y = 0 \ . \tag{4}$$

Please turn the page for questions f) and g).

f. Look for solutions of equation (4) of the form $u_y(y) = u(y)e^{-y/2s}$. Derive a second-order differential equation for $u(y)$ and solve this equation subject to the boundary conditions $u(0) = u(h) = 0$.

g. Show that the eigenfrequencies ω_n of the interchange modes are given by

$$\omega_n^2 = -\frac{g}{s}\frac{4k^2h^2s^2}{h^2 + 4s^2(k^2h^2 + n^2\pi^2)} \qquad n = 1, 2 \ldots \tag{5}$$

For which values of the scale parameter s are the interchange modes stable ?